Comptes Rendus
Mathématique
2020, 358, nº 11-12, p. 1231-1238
https://doi.org/10.5802/crmath.139

Probability Theory / *Probabilités*

On the Baum–Katz theorem for sequences of pairwise independent random variables with regularly varying normalizing constants

Lê Văn Thành [a]

[a] Department of Mathematics, Vinh University, 182 Le Duan, Vinh, Nghe An, Vietnam
E-mail: levt@vinhuni.edu.vn

Abstract. This paper proves the Baum–Katz theorem for sequences of pairwise independent identically distributed random variables with general norming constants under optimal moment conditions. The proof exploits some properties of slowly varying functions and the de Bruijn conjugates, and uses the techniques developed by Rio (1995) to avoid using the maximal type inequalities.

2020 Mathematics Subject Classification. 60F15.

Manuscript received 5th June 2020, revised 5th November 2020, accepted 7th November 2020.

1. Introduction and result

Let $1 \le p < 2, \alpha p \ge 1$ and $\{X, X_n, n \ge 1\}$ be a sequence of pairwise independent identically distributed (p.i.i.d.) random variables. In this paper, by using some results related to slowly varying functions and techniques developed by Rio [10], we provide the necessary and sufficient conditions for

$$\sum_n n^{\alpha p-2}\mathbb{P}\left(\max_{1\le k\le n}\left|\sum_{i=1}^k X_i\right| > \varepsilon n^{1/\alpha}\widetilde{L}(n^{1/\alpha})\right) < \infty \text{ for all } \varepsilon > 0, \tag{1}$$

where $\widetilde{L}(\cdot)$ is the de Bruijn conjugate of a slowly varying function $L(\cdot)$. The result provides the rate of convergence in the Marcinkiewicz–Zygmund strong law of large numbers (SLLN) with regularly varying normalizing constants. When the random variables are i.i.d. with $\mathbb{E}(X) = 0$, $\mathbb{E}(|X|^p) < \infty$, and $L(\cdot) = 1$, (1) was obtained by Baum and Katz [2].

The notion of regularly varying function can be found in Seneta [12, Chapter 1]. A real-valued function $R(\cdot)$ is said to be regularly varying with index of regular variation $\rho \in \mathbb{R}$ if it is a positive and measurable function on $[A, \infty)$ for some $A > 0$, and for each $\lambda > 0$,

$$\lim_{x\to\infty}\frac{R(\lambda x)}{R(x)} = \lambda^\rho.$$

A regularly varying function with the index of regular variation $\rho = 0$ is called to be slowly varying. It is well known that a function $R(\cdot)$ is regularly varying with the index of regular variation ρ if and only if it can be written in the form

$$R(x) = x^{\rho} L(x)$$

where $L(\cdot)$ is a slowly varying function (see, e.g., Seneta [12, p. 2]). Seneta [11] (see also in Bingham et al. [3, Lemma 1.3.2]) proved that if $L(\cdot)$ is a slowly varying function defined on $[A,\infty)$ for some $A > 0$, then there exists $B \geq A$ such that $L(x)$ is bounded on every finite closed interval $[a,b] \subset [B,\infty)$. Galambos and Seneta [7, p. 111] showed that for any slowly varying function $L(x)$, there exists a differentiable slowly varying function $L_1(\cdot)$ defined on $[B,\infty)$ for some $B \geq A$ such that

$$\lim_{x\to\infty} \frac{L(x)}{L_1(x)} = 1 \text{ and } \lim_{x\to\infty} \frac{xL_1'(x)}{L_1(x)} = 0.$$

Conversely, if $L(\cdot)$ is a positive differentiable function satisfying

$$\lim_{x\to\infty} \frac{xL'(x)}{L(x)} = 0, \tag{2}$$

then $L(\cdot)$ is a slowly varying function. If $L(\cdot)$ be a differentiable slowly varying function satisfying (2), then by direct calculations (see, e.g. [1, Lemma 2.3]), we can show that for all $p > 0$, there exists $B > 0$ such that

$$x^p L(x) \text{ is strictly increasing on } [B,\infty),\ x^{-p} L(x) \text{ is strictly decreasing on } [B,\infty). \tag{3}$$

Let $L(\cdot)$ be a slowly varying function. Then by [3, Theorem 1.5.13], there exists a slowly varying function $\widetilde{L}(\cdot)$, unique up to asymptotic equivalence, satisfying

$$\lim_{x\to\infty} L(x)\widetilde{L}(xL(x)) = 1 \text{ and } \lim_{x\to\infty} \widetilde{L}(x)L\big(x\widetilde{L}(x)\big) = 1. \tag{4}$$

The function $\widetilde{L}$ is called the de Bruijn conjugate of L, and $(L,\widetilde{L})$ is called a (slowly varying) conjugate pair (see, e.g., Bingham et al. [3, p. 29]). By [3, Proposition 1.5.14], if $(L,\widetilde{L})$ is a conjugate pair, then for $a,b,\alpha > 0$, each of $(L(ax),\widetilde{L}(bx))$, $(aL(x),a^{-1}\widetilde{L}(x))$, $((L(x^{\alpha}))^{1/\alpha},(\widetilde{L}(x^{\alpha}))^{1/\alpha})$ is a conjugate pair. Bojanić and Seneta [4] (see also [3, Theorem 2.3.3 and Corollary 2.3.4] in Bingham et al.) proved that if $L(\cdot)$ is a slowly varying function satisfying

$$\lim_{x\to\infty} \left(\frac{L(\lambda_0 x)}{L(x)} - 1\right) \log(L(x)) = 0, \tag{5}$$

for some $\lambda_0 > 1$, then for all $\alpha \in \mathbb{R}$,

$$\lim_{x\to\infty} \frac{L(xL^{\alpha}(x))}{L(x)} = 1, \tag{6}$$

and therefore, we can choose (up to aymptotic equivalence) $\widetilde{L}(x) = 1/L(x)$. Especially, if for some $\gamma \in \mathbb{R}$, $L(x) = \log^{\gamma}(x+2)$, $x \geq 0$, then $\widetilde{L}(x) = 1/L(x)$. For $\alpha,\beta > 0$ and for $f(x) = x^{\beta/\alpha}L^{1/\alpha}(x^{\beta})$, $g(x) = x^{\alpha/\beta}\widetilde{L}^{\beta}(x^{\alpha})$, we have (see [3, Theorem 1.5.12 and Proposition 1.5.15])

$$\lim_{x\to\infty} \frac{f(g(x))}{x} = \lim_{x\to\infty} \frac{g(f(x))}{x} = 1. \tag{7}$$

Here and thereafter, for a slowly varying function $L(\cdot)$, we denote the de Bruijn conjugate of $L(\cdot)$ by $\widetilde{L}(\cdot)$. We will assume, without loss of generality, that $L(x)$ and $\widetilde{L}(x)$ are both defined on $[0,\infty)$ and differentiable on $[A,\infty)$ for some $A > 0$.Theorem 1 is the main result of this paper. The Marcinkiewicz–Zygmund SLLN with regularly varying normalizing constants was also studied recently by Anh et al. [1], where the proof is based on the Kolmogorov maximal inequality.

Theorem 1. *Let $1 \leq p < 2$, and let $\{X, X_n, n \geq 1\}$ be a sequence of p. i.i.d. random variables, $L(\cdot)$ a slowly varying function defined on $[0,\infty)$. When $p = 1$, we assume further that $L(x) \geq 1$ and is increasing on $[0,\infty)$. Then the following statements are equivalent.*

(i) *The random variable X satisfies*

$$\mathbb{E}(X) = 0,\ \mathbb{E}\left(|X|^p L^p(|X|)\right) < \infty. \tag{8}$$

(ii) *For all $\alpha \geq 1/p$, we have*

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{P}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right| > \varepsilon n^{\alpha}\widetilde{L}\left(n^{\alpha}\right)\right) < \infty \text{ for all } \varepsilon > 0. \tag{9}$$

(iii) *The Marcinkiewicz–Zygmund-type SLLN*

$$\lim_{n\to\infty} \frac{\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|}{n^{1/p}\widetilde{L}\left(n^{1/p}\right)} = 0 \ \text{ almost surely (a.s.)} \tag{10}$$

holds.

For a sequence of random variables which are pairwise independent but not identically distributed, Csörgő et al. [5] proved that the Kolmogorov condition alone does not ensure the SLLN. On the case where the random variables $\{X, X_n, n \geq 1\}$ are p. i.i.d, Etemadi [6] proved that the Kolmogorov SLLN holds under moment condition $\mathbb{E}(|X|) < \infty$. For $\gamma > 0$, Martikainen [9] proved that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n\log^{-\gamma}(n)} = 0 \ \text{ a.s.}$$

if and only if $\mathbb{E}(X) = 0$ and $\mathbb{E}(|X|\log^{\gamma}(|X|+2)) < \infty$. This is a special case of Theorem 1 when $p = 1$ and $L(x) \equiv \log^{\gamma}(x+2)$. For the case where $1 < p < 2$, Martikainen [8] proved that if $\mathbb{E}(X) = 0$ and $\mathbb{E}(|X|^p \log^r(|X|+1)) < \infty$ for some $r > \max\{0, 4p-6\}$, then the Marcinkiewicz–Zygmund SLLN holds. By letting $L(x) \equiv 1$, we obtain the following corollary. When $\alpha = 1/p$, this corollary was obtained by Rio [10].

Corollary 2. *Let $1 \leq p < 2$, and let $\{X, X_n, n \geq 1\}$ be a sequence of p. i.i.d. random variables. Then the following statements are equivalent.*

(i) *The random variable X satisfies*

$$\mathbb{E}(X) = 0,\ \mathbb{E}\left(|X|^p\right) < \infty.$$

(ii) *For all $\alpha \geq 1/p$, we have*

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{P}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right| > \varepsilon n^{\alpha}\right) < \infty \text{ for all } \varepsilon > 0.$$

(iii) *The Marcinkiewicz–Zygmund SLLN*

$$\lim_{n\to\infty} \frac{\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|}{n^{1/p}} = 0 \ \text{ a.s.}$$

holds.

2. Proof

To prove the main result, we firstly introduce some preliminaries. Through this paper, $C(\cdot)$, $C_1(\cdot), C_2(\cdot), \ldots$ denote constants which depend only on variables appearing in the parentheses. Lemma 3 is a direct consequence of Karamata's theorem (see [3]).

Lemma 3. *Let $a, b > 1$, and $L(\cdot)$ be a differentiable slowly varying function defined on $[0,\infty)$. Then*

$$\sum_{k=1}^{n} a^k L\left(b^k\right) \leq C_1(a,b)a^n L\left(b^n\right).$$

Lemma 4 gives simple criterions for $\mathbb{E}(|X|^p L^p(|X|)) < \infty$. When $\alpha = 1/p$, the equivalence of (11) and (12) was established by Anh et al. [1, Proposition 2.6].

Lemma 4. *Let $p \ge 1$, $\alpha p \ge 1$, and X be a random variable. Let $L(x)$ be a slowly varying function defined on $[0,\infty)$, and $b_n = n^\alpha \widetilde{L}(n^\alpha)$, $n \ge 1$. Assume that $x^{1/\alpha} L^{1/\alpha}(x)$ and $x^\alpha \widetilde{L}(x^\alpha)$ are strictly increasing on $[A,\infty)$ for some $A > 0$. Then the following statements are equivalent.*

$$\mathbb{E}\left(|X|^p L^p(|X|)\right) < \infty. \tag{11}$$

$$\sum_{n=1}^{\infty} n^{\alpha p-1} \mathbb{P}\left(|X| > b_n\right) < \infty. \tag{12}$$

$$\sum_{n=1}^{\infty} 2^{n\alpha p} \mathbb{P}\left(b_{2^{n-1}} < |X| \le b_{2^n}\right) < \infty. \tag{13}$$

Proof. Let $f(x) = x^{1/\alpha} L^{1/\alpha}(x)$, $g(x) = x^\alpha \widetilde{L}(x^\alpha)$. By using (7) with $\beta = 1$, we have

$$f\left(g(x)\right) \sim g\left(f(x)\right) \sim x \text{ as } x \to \infty. \tag{14}$$

Firstly, we will prove (11) is equivalent to (12). For a non negative random variable Y and $r > 0$, $\mathbb{E}Y^r < \infty$ if and only if $\sum_{n=1}^{\infty} n^{r-1}\mathbb{P}(Y > n) < \infty$. Applying this, we have that $\mathbb{E}(f^{\alpha p}(|X|)) < \infty$ if and only if

$$\sum_{n=1}^{\infty} n^{\alpha p-1} \mathbb{P}\left(f(|X|) > n\right) < \infty. \tag{15}$$

Combining (14) with the assumption that $f(x)$ and $g(x)$ are strictly increasing on $[A,\infty)$, we see that (15) is equivalent to

$$\sum_{n=1}^{\infty} n^{\alpha p-1} \mathbb{P}\left(|X| > n^\alpha \widetilde{L}\left(n^\alpha\right)\right) < \infty.$$

The proof of the equivalence of (11) and (12) is completed. Now, we will prove (11) is equivalent to (13). For n large enough, on event $(b_{2^{n-1}} < |X| \le b_{2^n})$, we have

$$f^{\alpha p}\left(b_{2^{n-1}}\right) < f^{\alpha p}(|X|) \le f^{\alpha p}\left(b_{2^n}\right),$$

or equivalently,

$$\left(f\left(g\left(2^{n-1}\right)\right)\right)^{\alpha p} < |X|^p L^p\left(|X|\right) \le \left(f\left(g\left(2^n\right)\right)\right)^{\alpha p}. \tag{16}$$

Combining (14) and (16), we see that (13) is equivalent to (11). □

Proof of Theorem 1. By the arguments leading to (2) and (3), without loss of generality, we can assume that there exists a positive integer A large enough such that $x^{1/\alpha} L(x^{1/\alpha})$, $x^\alpha \widetilde{L}(x^\alpha)$ and $x^{p-1} L^p(x)$ (for $p > 1$) are strictly increasing on $[A,\infty)$.

Firstly, we prove the implication ((i) ⇒ (ii)). It is easy to see that (9) is equivalent to

$$\sum_{n=1}^{\infty} 2^{n(\alpha p-1)} \mathbb{P}\left(\max_{1\le j<2^n} \left|\sum_{i=1}^{j} X_i\right| > \varepsilon 2^{n\alpha} \widetilde{L}\left(2^{n\alpha}\right)\right) < \infty \text{ for all } \varepsilon > 0. \tag{17}$$

Assume that (8) holds. For $n \ge 1$, set $b_n = n^\alpha \widetilde{L}(n^\alpha)$ and

$$X_{i,n} = X_i \mathbf{1}\left(|X_i| \le b_n\right),\ 1 \le i \le n.$$

For all $\varepsilon > 0$ and $n \ge 1$, we have

$$\begin{aligned}\mathbb{P}\left(\max_{1\le j<2^n} \left|\sum_{i=1}^{j} X_i\right| > \varepsilon b_{2^n}\right) &\le \mathbb{P}\left(\max_{1\le i<2^n} |X_i| > b_{2^n}\right) + \mathbb{P}\left(\max_{1\le j<2^n} \left|\sum_{i=1}^{j} X_{i,2^n}\right| > \varepsilon b_{2^n}\right)\\ &\le \mathbb{P}\left(\max_{1\le i<2^n} |X_i| > b_{2^n}\right) + \mathbb{P}\left(\max_{1\le j<2^n} \left|\sum_{i=1}^{j} (X_{i,2^n} - \mathbb{E}X_{i,2^n})\right| > \varepsilon b_{2^n} - \sum_{i=1}^{2^n} \left|E(X_{i,2^n})\right|\right).\end{aligned} \tag{18}$$

Since $\widetilde{L}(\cdot)$ is slowly varying, $b_{2^{n+1}} \le 4^{\alpha} b_{2^n}$ for $n \ge n_0$ for some $n_0 \ge A$. By the second half of (8) and Lemma 4, we have

$$\begin{aligned}\infty > \sum_{j=1}^{\infty} j^{\alpha p-1}\mathbb{P}\left(4^{\alpha}|X| > b_j\right) &= \sum_{j=1}^{\infty} j^{\alpha p-2}\sum_{i=1}^{j}\mathbb{P}\left(4^{\alpha}|X_i| > b_j\right)\\ \ge \sum_{j=1}^{\infty} j^{\alpha p-2}\mathbb{P}\left(\max_{1\le i\le j} 4^{\alpha}|X_i| > b_j\right) &= \sum_{n=0}^{\infty}\sum_{j=2^n}^{2^{n+1}-1} j^{\alpha p-2}\mathbb{P}\left(\max_{1\le i\le j} 4^{\alpha}|X_i| > b_j\right)\\ \ge \frac{1}{2}\sum_{n=0}^{\infty}\sum_{j=2^n}^{2^{n+1}-1} 2^{n(\alpha p-2)}\mathbb{P}\left(\max_{1\le i<2^n} 4^{\alpha}|X_i| > b_j\right)&\\ \ge \frac{1}{2}\sum_{n=n_0}^{\infty} 2^{n(\alpha p-1)}\mathbb{P}\left(\max_{1\le i<2^n} |X_i| > b_{2^n}\right).&\end{aligned}\tag{19}$$

For $n \ge 1$, the first half of (8) imply that

$$\begin{aligned}\frac{\left|\sum_{i=1}^{n}\mathbb{E}\left(X_{i,n}\right)\right|}{b_n} &\le \frac{\sum_{i=1}^{n}\left|\mathbb{E}X_i\mathbf{1}\left(|X_i| > b_n\right)\right|}{b_n}\\ &\le \frac{n\mathbb{E}\left(|X|\mathbf{1}\left(|X| > b_n\right)\right)}{b_n}.\end{aligned}\tag{20}$$

For n large enough and for $\omega \in (|X| > b_n)$, we have

$$\begin{aligned}\frac{n}{b_n} \le \frac{n^{(p-1)\alpha}\widetilde{L}^{p-1}(n^{\alpha})}{\widetilde{L}^{p}(n^{\alpha})} &= \frac{\left(n^{\alpha}\widetilde{L}(n^{\alpha})\right)^{p-1}L^{p}\left(n^{\alpha}\widetilde{L}(n^{\alpha})\right)}{\widetilde{L}^{p}(n^{\alpha})L^{p}\left(n^{\alpha}\widetilde{L}(n^{\alpha})\right)}\\ \le 2b_n^{p-1}L^{p}(b_n) &\le 2|X(\omega)|^{p-1}L^{p}(|X(\omega)|),\end{aligned}\tag{21}$$

where we have applied the second half of (4) in the second inequality and the monotonicity of $x^{p-1}L^{p}(x)$ in the third inequality. It follows from (21) and the second half of (8) that

$$\frac{n\left|\mathbb{E}\left(|X|\mathbf{1}\left(|X| > b_n\right)\right)\right|}{b_n} \le 2\mathbb{E}\left(|X|^{p}L^{p}(|X|)\mathbf{1}\left(|X| > b_n\right)\right) \to 0 \text{ as } n \to \infty.\tag{22}$$

From (18), (19), (20) and (22), the proof of (17) will be completed if we can show that

$$\sum_{n=1}^{\infty} 2^{n(\alpha p-1)}\mathbb{P}\left(\max_{1\le j<2^n}\left|\sum_{i=1}^{j}\left(X_{i,2^n} - \mathbb{E}X_{i,2^n}\right)\right| \ge \varepsilon b_{2^{n-1}}\right) < \infty \text{ for all } \varepsilon > 0.\tag{23}$$

For $m \ge 0$, set $S_{0,m} = 0$ and

$$S_{j,m} = \sum_{i=1}^{j}\left(X_{i,2^m} - \mathbb{E}X_{i,2^m}\right),\ j \ge 1.$$

Now, we use techniques developed by Rio [10]. For $1 \le j < 2^n$ and for $0 \le m \le n$, let $k = \lfloor j/2^m \rfloor$ be the greatest integer which is less than or equal to $j/2^m$. Then $0 \le k < 2^{n-m}$ and $k2^m \le j < (k+1)2^m$. Let $j_m = k2^m$, then

$$S_{j,n} = \sum_{m=1}^{n}\left(S_{j_{m-1},m-1} - S_{j_m,m-1}\right) + \sum_{m=1}^{n}\left(S_{j,m} - S_{j,m-1} - S_{j_m,m} + S_{j_m,m-1}\right),\ 1 \le j < 2^n.\tag{24}$$

and

$$\left|S_{j,m} - S_{j,m-1} - S_{j_m,m} + S_{j_m,m-1}\right| \le \sum_{i=j_m+1}^{j_m+2^m}\left(\left|X_{i,2^m} - X_{i,2^{m-1}}\right| + \mathbb{E}\left|X_{i,2^m} - X_{i,2^{m-1}}\right|\right).\tag{25}$$

Set

$$Y_{i,m} = \left|X_{i,2^m} - X_{i,2^{m-1}}\right| - \mathbb{E}\left(\left|X_{i,2^m} - X_{i,2^{m-1}}\right|\right),\ m \ge 1, i \ge 1.$$

It follows from (25) that

$$\left|S_{j,m}-S_{j,m-1}-S_{j_m,m}+S_{j_m,m-1}\right|\le\sum_{i=j_m+1}^{j_m+2^m}Y_{i,m}+2^{m+1}\mathbb{E}\left(\left|X_{1,2^m}-X_{1,2^{m-1}}\right|\right). \tag{26}$$

By the definition of j_m, we have either $j_{m-1}=j_m$ or $j_{m-1}=j_m+2^{m-1}$. Therefore

$$\left|S_{j_{m-1},m-1}-S_{j_m,m-1}\right|\le\left|\sum_{i=j_m+1}^{j_m+2^{m-1}}\left(X_{i,2^{m-1}}-\mathbb{E}(X_{i,2^{m-1}})\right)\right|. \tag{27}$$

Combining (24), (26) and (27), we have

$$\begin{aligned}\max_{1\le j<2^n}\left|S_{j,n}\right|\le&\sum_{m=1}^{n}\max_{0\le k<2^{n-m}}\left|\sum_{i=k2^m+1}^{k2^m+2^{m-1}}\left(X_{i,2^{m-1}}-\mathbb{E}\left(X_{i,2^{m-1}}\right)\right)\right|\\&+\sum_{m=1}^{n}\max_{0\le k<2^{n-m}}\left|\sum_{i=k2^m+1}^{(k+1)2^m}Y_{i,m}\right|+\sum_{m=1}^{n}2^{m+1}\mathbb{E}\left(\left|X_{1,2^m}-X_{1,2^{m-1}}\right|\right).\end{aligned} \tag{28}$$

It follows from (22) that

$$\frac{2^m\mathbb{E}\left(|X|\mathbf{1}\left(|X|>b_{2^{m-1}}\right)\right)}{b_{2^m}}\to0\ \text{ as }\ m\to\infty. \tag{29}$$

From (29) and Lemma 3, we can apply Toeplitz's lemma and conclude that

$$\lim_{n\to\infty}\frac{\sum_{m=1}^{n}2^m\mathbb{E}\left(|X|\mathbf{1}\left(|X|>b_{2^{m-1}}\right)\right)}{b_{2^n}}=0. \tag{30}$$

By using

$$\mathbb{E}\left(\left|X_{1,2^m}-X_{1,2^{m-1}}\right|\right)\le\mathbb{E}\left(|X|\mathbf{1}\left(|X|>b_{2^{m-1}}\right)\right),\ m\ge A,$$

we have from (30) that

$$\lim_{n\to\infty}\frac{\sum_{m=1}^{n}2^{m+1}\mathbb{E}\left(\left|X_{1,2^m}-X_{1,2^{m-1}}\right|\right)}{b_{2^n}}=0. \tag{31}$$

Let $\varepsilon_1>0$ be arbitrary, and let a and b be positive constants satisfying

$$\alpha p/2<a<\alpha,\ a+b=\alpha.$$

For $n\ge1$, $0\le m\le n$, let $\lambda_{m,n}=\varepsilon_1 2^{bm}2^{an}\widetilde{L}(2^{n\alpha})$. Then

$$\begin{aligned}\sum_{m=1}^{n}\lambda_{m,n}&=\varepsilon_1 2^{an}\widetilde{L}\left(2^{n\alpha}\right)\sum_{m=1}^{n}2^{mb}\\&=\varepsilon_1 2^{an}2^b\widetilde{L}\left(2^{n\alpha}\right)\frac{2^{bn}-1}{2^b-1}\le\frac{2^b\varepsilon_1 b_{2^n}}{2^b-1}:=C_1(b)\varepsilon_1 b_{2^n}.\end{aligned} \tag{32}$$

By (32) and Chebyshev's inequality, we have

$$\begin{aligned}
\mathbb{P}&\left(\sum_{m=1}^{n}\max_{0\le k<2^{n-m}}\left|\sum_{i=k2^m+1}^{(k+1)2^m}Y_{i,m}\right|\ge C_1(b)\varepsilon_1 b_{2^n}\right)\\
&\le\sum_{m=1}^{n}\mathbb{P}\left(\max_{0\le k<2^{n-m}}\left|\sum_{i=k2^m+1}^{(k+1)2^m}Y_{i,m}\right|\ge\lambda_{m,n}\right)\\
&\le\sum_{m=1}^{n}\lambda_{m,n}^{-2}\mathbb{E}\left(\max_{0\le k<2^{n-m}}\left|\sum_{i=k2^m+1}^{(k+1)2^m}Y_{i,m}\right|\right)^2\\
&\le\sum_{m=1}^{n}\lambda_{m,n}^{-2}\sum_{k=0}^{2^{n-m}-1}\mathbb{E}\left(\sum_{i=k2^m+1}^{(k+1)2^m}Y_{i,m}\right)^2\\
&\le\sum_{m=1}^{n}2^n\lambda_{m,n}^{-2}\mathbb{E}\left(\left|X_{i,2^m}-X_{i,2^{m-1}}\right|^2\right)\\
&\le\sum_{m=1}^{n}2^{n+1}\lambda_{m,n}^{-2}\left(\mathbb{E}\left(X_{i,2^m}^2\right)+\mathbb{E}\left(X_{i,2^{m-1}}^2\right)\right)\\
&=\sum_{m=1}^{n}2^{n+1}\lambda_{m,n}^{-2}\left(\mathbb{E}\left(X^2\mathbf{1}\left(|X|\le b_{2^m}\right)\right)+\mathbb{E}\left(X^2\mathbf{1}\left(|X|\le b_{2^{m-1}}\right)\right)\right),
\end{aligned}\tag{33}$$

and

$$\begin{aligned}
\mathbb{P}&\left(\sum_{m=1}^{n}\max_{0\le k<2^{n-m}}\left|\sum_{i=k2^m+1}^{k2^m+2^{m-1}}\left(X_{i,2^{m-1}}-\mathbb{E}\left(X_{i,2^{m-1}}\right)\right)\right|\ge C_1(b)\varepsilon_1 b_{2^n}\right)\\
&\le\sum_{m=1}^{n}\mathbb{P}\left(\max_{0\le k<2^{n-m}}\left|\sum_{i=k2^m+1}^{k2^m+2^{m-1}}\left(X_{i,2^{m-1}}-\mathbb{E}\left(X_{i,2^{m-1}}\right)\right)\right|\ge\lambda_{m,n}\right)\\
&\le\sum_{m=1}^{n}\lambda_{m,n}^{-2}\mathbb{E}\left(\max_{0\le k<2^{n-m}}\left|\sum_{i=k2^m+1}^{k2^m+2^{m-1}}\left(X_{i,2^{m-1}}-\mathbb{E}\left(X_{i,2^{m-1}}\right)\right)\right|\right)^2\\
&\le\sum_{m=1}^{n}\lambda_{m,n}^{-2}\sum_{k=0}^{2^{n-m}-1}\mathbb{E}\left(\sum_{i=k2^m+1}^{k2^m+2^{m-1}}\left(X_{i,2^{m-1}}-\mathbb{E}\left(X_{i,2^{m-1}}\right)\right)\right)^2\\
&\le\sum_{m=1}^{n}2^n\lambda_{m,n}^{-2}\mathbb{E}\left(X_{i,2^{m-1}}^2\right)=\sum_{m=1}^{n}2^n\lambda_{m,n}^{-2}\mathbb{E}\left(X^2\mathbf{1}\left(|X|\le b_{2^{m-1}}\right)\right).
\end{aligned}\tag{34}$$

Since $\alpha p<2a$ and $1\le p<2$, elementary calculations show that

$$\sum_{n=1}^{\infty}2^{n(\alpha p-1)}\sum_{m=1}^{n}2^n\lambda_{m,n}^{-2}\le\frac{1}{\varepsilon_1^2(4^b-1)}\sum_{n=1}^{\infty}2^{n(\alpha p-2a)}\widetilde{L}^{-2}\left(2^{n\alpha}\right)<\infty.\tag{35}$$

We recall that b_n is strictly increasing on $[A,\infty)$. From (28), (31), and (33)–(35), the proof of (23) is completed if we can show that

$$I:=\sum_{n=A}^{\infty}2^{n(\alpha p-1)}\sum_{m=A}^{n}2^n\lambda_{m,n}^{-2}\sum_{k=A}^{m}b_{2^k}^2\mathbb{P}\left(b_{2^{k-1}}<|X|\le b_{2^k}\right)<\infty.\tag{36}$$

By using the second half of (8), Lemmas 3–4, and definition of a and b, we have

$$\begin{aligned}
I = \sum_{n=A}^{\infty} 2^{n(\alpha p-1)}\varepsilon_1^{-2} 2^{n(1-2a)}\widetilde{L}^{-2}\left(2^{n\alpha}\right)\left(\sum_{m=A}^{n} 2^{-2mb}\sum_{k=A}^{m} b_{2^k}^2 \mathbb{P}\left(b_{2^{k-1}} < |X| \le b_{2^k}\right)\right)& \\
\le C_2(b)\varepsilon_1^{-2}\sum_{n=A}^{\infty} 2^{n(\alpha p-2a)}\widetilde{L}^{-2}\left(2^{n\alpha}\right)\sum_{k=A}^{n} 2^{-2kb} b_{2^k}^2 \mathbb{P}\left(b_{2^{k-1}} < |X| \le b_{2^k}\right)& \\
\le C_2(b)\varepsilon_1^{-2}\sum_{k=A}^{\infty}\left(\sum_{n=k}^{\infty} 2^{n(\alpha p-2a)}\widetilde{L}^{-2}\left(2^{n\alpha}\right)\right)2^{-2kb} b_{2^k}^2 \mathbb{P}\left(b_{2^{k-1}} < |X| \le b_{2^k}\right)& \\
\le C(\alpha,a,b,p)\varepsilon_1^{-2}\sum_{k=A}^{\infty} 2^{k(\alpha p-2a)}\widetilde{L}^{-2}(2^{k\alpha})2^{-2kb} b_{2^k}^2 \mathbb{P}\left(b_{2^{k-1}} < |X| \le b_{2^k}\right)& \\
= C(\alpha,a,b,p)\varepsilon_1^{-2}\sum_{k=A}^{\infty} 2^{k\alpha p}\mathbb{P}\left(b_{2^{k-1}} < |X| \le b_{2^k}\right) < \infty&
\end{aligned}$$

thereby proving (36). The proof of the implication ((i) $\Rightarrow$ (ii)) is completed.

By choosing $\alpha = 1/p$, we have the implication ((ii) $\Rightarrow$ (iii)). The proof of the implication ((iii) $\Rightarrow$(i)) follows from the Borel–Cantelli lemma for pairwise independent events and Lemma 4 (see [1, the proof of Theorem 3.1]). $\square$

References

[1] V. T. N. Anh, N. T. T. Hien, L. V. Thành, V. T. H. Van, "The Marcinkiewicz–Zygmund-type strong law of large numbers with general normalizing sequences", *J. Theor. Probab.* (2019), p. 1-18.

[2] L. E. Baum, M. Katz, "Convergence rates in the law of large numbers", *Trans. Am. Math. Soc.* **120** (1965), no. 1, p. 108-123.

[3] N. H. Bingham, C. M. Goldie, J. L. Teugels, *Regular variation*, paperback ed. ed., Encyclopedia of Mathematics and Its Applications, vol. 27, Cambridge University Press, 1989.

[4] R. Bojanic, E. Seneta, "Slowly varying functions and asymptotic relations", *J. Math. Anal. Appl.* **34** (1971), no. 2, p. 302-315.

[5] S. Csörgő, K. Tandori, V. Totik, "On the strong law of large numbers for pairwise independent random variables", *Acta Math. Hung.* **42** (1983), no. 3-4, p. 319-330.

[6] N. Etemadi, "An elementary proof of the strong law of large numbers", *Z. Wahrscheinlichkeitstheor. Verw. Geb.* **55** (1981), no. 1, p. 119-122.

[7] J. Galambos, E. Seneta, "Regularly varying sequences", *Proc. Am. Math. Soc.* **41** (1973), no. 1, p. 110-116.

[8] A. Martikainen, "On the strong law of large numbers for sums of pairwise independent random variables", *Stat. Probab. Lett.* **25** (1995), no. 1, p. 21-26.

[9] A. I. Martikaĭnen, "A remark on the strong law of large numbers for sums of pairwise independent random variables", *J. Math. Sci.* **75** (1995), no. 5, p. 1944-1946.

[10] E. Rio, "Vitesses de convergence dans la loi forte pour des suites dépendantes (Rates of convergence in the strong law for dependent sequences)", *C. R. Acad. Sci. Paris* **320** (1995), no. 4, p. 469-474.

[11] E. Seneta, "An interpretation of some aspects of Karamata's theory of regular variation", *Publ. Inst. Math.* **15** (1973), no. 29, p. 111-119.

[12] ———, *Regularly varying functions*, Lecture Notes in Mathematics, vol. 508, Springer, 1976.